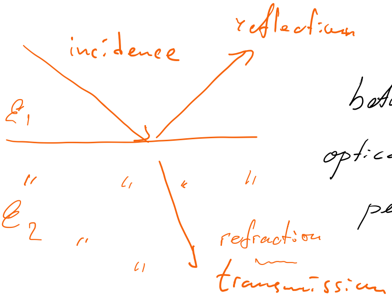
Light at the border of two dielectrics

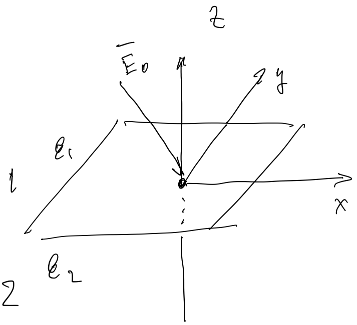


When the light going through the interface between two media of different optical qualities, it partially penetrates inside second media and partially reflect. Our goal in this chapter is to derive famous laws of geometrical optics based on Maxwell's e/m theory, as well as answer on polarization state of all the field and intensities

We will answer:

1) direction of reflected and refracted wave
2) intensity of waves
3) polarization of waves

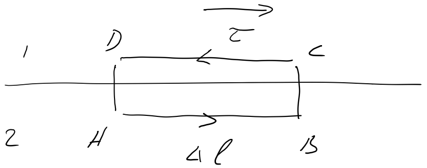
It is import to note here border conditions, that both reflected and refracted waves experience.



$$\begin{cases} E_{1\tau} = E_{2\tau} \\ B_{1\tau} = B_{2\tau} \end{cases}$$

This is very easy to show.

Let's take contour ABCD:



The borders of this contour go at different media.

We will recall integrated Maxwell's equation for $\bar{E}$ field in contour:

$$\oint_{l} \bar{E}\, d\bar{l} = -\frac{\partial}{\partial t} \int_{S} \bar{B}\, d\bar{S} \quad (1)$$

Now, let's make $AD \to 0$ and $CB \to 0$.
In this case, circulation of $\bar{E}$ will be:

$$\oint_{l} \bar{E}\, d\bar{l} \to \Delta E_{1\tau} \Delta l - \Delta E_{2\tau} \Delta l.$$

$E_{1\tau}$ and $E_{2\tau}$ are projections of E_1 and E_2 to vector $\bar{\tau}$ that is parallel to the border, so-called tangential.
The flux on the right side will be zero:

$$\int_{S} \bar{B}\, d\bar{S} = 0$$

The same can be easily extended to:

$$c^2 \oint_{l} \bar{B}\, d\bar{l} = \frac{\partial}{\partial t} \int_{S} \bar{E}\, d\bar{s}$$

Thus

$$E_{1\tau} = E_{2\tau}$$

$$B_{1\tau} = B_{2\tau}$$

From other Maxwell's equations, we can derive the normal components:

$$\begin{cases} \bar{\nabla} \bar{E} = 0 \\ c^2 [\bar{\nabla} \bar{B}] = \frac{\partial E}{\partial t} \end{cases}$$

Please, try yourself

We acquired two important border conditions, that express continuity of electric and magnetic vectors of the field on the border:

$$\begin{cases} E_{1n} = E_{2n} \\ B_{1n} = B_{2n} \end{cases}$$

It is important to notice, that existence of single refracted wave in second medium does not directly follows from Maxwell's equations. It is based on additional assumption, so-called radiation condition. For example, border conditions will be conserved even for two waves propagating in medium. One can propagate to the border, while another from. However, it will give different results.

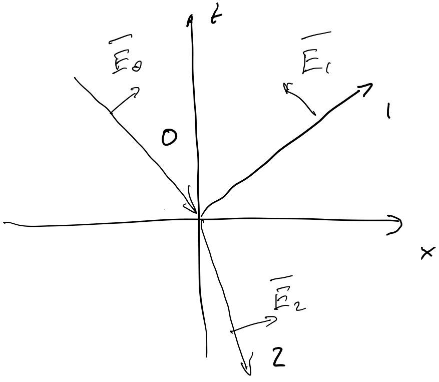
Radiation condition is based on principle of causality. This principle gives us criterion that results in solutions that have physical meaning:

Excited body can create waves that only propagates away.

In case of our problem of light on the border of two dielectrics the only solution that makes physical sense is in case, where there are only three waves - incident, reflected and refracted.

Let's assume we have incoming monochromatic wave:

$$\begin{cases} \bar{E} = \bar{E}_0 e^{i(\omega t - \bar{k}\bar{r})} \\ B = \bar{B}_0 e^{i(\omega t - \bar{k}\bar{r})} \end{cases}$$



$$\begin{cases} \bar E_0 = \bar E_{00}\, e^{i(\omega_0 t - \bar k_0 \bar r)} \\ \bar E_1 = \bar E_{01}\, e^{i(\omega_1 t - \bar k_1 \bar r)} \\ \bar E_2 = \bar E_{02}\, e^{i(\omega_2 t - \bar k_2 \bar r)} \end{cases}$$

It is not yet clear for us $\begin{cases} \omega_0 = \omega_1 = \omega_2 \\ k_0 = k_1 = k_2 \end{cases}$

Let's find it out!

Since $E_{1\tau} = E_{2\tau}$, then

$$(\bar E_0 + \bar E_1)_\tau = (\bar E_2)_\tau$$

$$E_{00\tau}\, e^{i(\omega_0 t - \bar k_0 \bar r)} + E_{01\tau}\, e^{i(\omega_1 t - \bar k_1 \bar r)} = E_{02\tau}\, e^{i(\omega_2 t - \bar k_2 \bar r)}$$

This should stay true at $\forall\, t$ and $\forall\, \bar r$.

Thus, let's fix $\bar{r}$:

$$Ae^{i\omega_0 t} + Be^{i\omega_1 t} = Ce^{i\omega_2 t} \quad (1)$$

A, B and C are time independent constants.

Let's differentiate:

$$i\omega_0 A e^{i\omega_0 t} + i\omega_1 B e^{i\omega_1 t} = i\omega_2 C e^{i\omega_2 t}$$

$$\frac{1}{\omega_2}\left(\omega_0 A e^{i\omega_0 t} + \omega_1 B e^{i\omega_1 t}\right) = Ce^{i\omega_2 t} \quad (2)$$

Let's subtract (1)-(2)

$$A(\omega_2-\omega_0)e^{i\omega_0 t} + B(\omega_2-\omega_1)e^{i\omega_1 t} = 0$$

Thus,

$$\boxed{\omega_0=\omega_1=\omega_2}$$

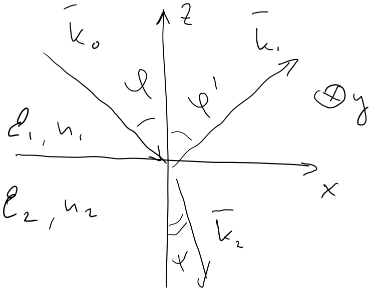
Crossing border of two dielectrics does not change frequency.

The same logic can be applied to if to take derivative over x:

$$Ae^{-ik_{0x}X} + Be^{-ik_{1x}X} = Ce^{-ik_{2x}X}$$

Please, derive

$$\begin{cases} k_{0x} = k_{1x} = k_{2x} \\ k_{0y} = k_{1y} = k_{2y} \end{cases}$$



$$\begin{cases} \omega_1 = \omega_2 = \omega_3 \\ k_{0x} = k_{1x} = k_{2x} \\ k_{0y} = k_{1y} = k_{2y} \end{cases}$$

Now, let's derive from border conditions the laws of geometrical optics.

① All beams are in one plane.

If $k_{0y} = 0 \Rightarrow k_{1y} = k_{2y} = 0$

②

$$k_{0x} = k_{1x}$$

$$k_0 \sin\varphi = k_1 \sin\varphi'$$

$$\frac{\omega}{v_1} \sin\varphi = \frac{\omega}{v_2} \sin\varphi' \Rightarrow \varphi = \varphi'$$

③

$$k_{0x} = k_{2x}$$

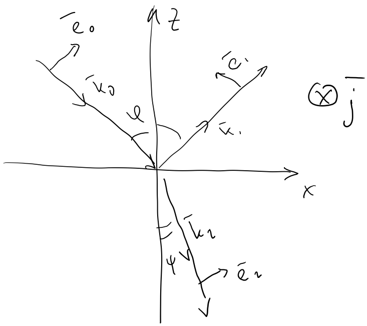
$$\frac{\omega}{v_1} \sin\varphi = \frac{\omega}{v_2} \sin\psi$$

$$\frac{\sin\varphi}{\sin\psi} = \frac{v_1}{v_2} = \frac{n_2}{n_1} \longrightarrow \text{Snell's law.}$$

Let's see how different polarization components reflect and refract – amplitude and polarization

Now, let's use superposition principle. We will look at each component as ∥ + ⊥.

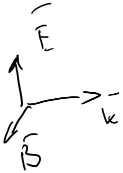
$$\bar E_0 = \bar E_0^{\parallel} + \bar E_0^{\perp}$$



$$\bar E_0^{\parallel} = \bar e_0 E_0^{\parallel}$$

$$\bar E_0^{\perp} = \bar j E_0^{\perp}$$

⊥ means polarization, i.e. E ⊥ plane of incidence:

1) 

2) $kE = \omega B \Rightarrow B = \frac{E}{v} = \frac{n}{c} E$

Why "–"?

$$\bar B_0^{\perp} = \bar e_0 B_0^{\perp} = \bar e_0 \frac{n_1}{c} E_0^{\perp}$$

$$\bar B_0^{\parallel} = -\bar j B_0^{\parallel} = -\bar j \frac{n_1}{c} E_0^{\parallel}$$

Similar

$$\bar E_1 = \bar E_1^{\parallel} + \bar E_1^{\perp} \qquad \bar E_2 = \bar E_2^{\parallel} + \bar E_2^{\perp}$$

$$\begin{cases} \bar{E}_1^{\parallel} = \bar{e}_1 E_1^{\parallel} \\ \bar{E}_1^{\perp} = \bar{j} E_1^{\perp} \\ \bar{B}_1^{\parallel} = -\bar{j}\frac{n_1}{c} E_1^{\parallel} \\ \bar{B}_1^{\perp} = \bar{e}_1 \frac{n_1}{c} E_1^{\perp} \end{cases} \qquad \begin{cases} \bar{E}_2^{\parallel} = \bar{e}_2 E_2^{\parallel} \\ \bar{E}_2^{\perp} = \bar{j} E_2^{\perp} \\ \bar{B}_2^{\parallel} = -\bar{j}\frac{n_2}{c} E_2^{\parallel} \\ \bar{B}_2^{\perp} = \bar{e}_2 \frac{n_2}{c} E_2^{\perp} \end{cases}$$

Let's remember border conditions

$$\begin{cases} \left(\bar{E}_0^{\parallel} + \bar{E}_1^{\parallel}\right)_x = \left(\bar{E}_2^{\parallel}\right)_x \\ \left(\bar{E}_0^{\perp} + \bar{E}_1^{\perp}\right)_y = \left(\bar{E}_2^{\perp}\right)_y \\ \left(\bar{B}_0^{\parallel} + \bar{B}_1^{\parallel}\right)_y = \left(\bar{B}_2^{\parallel}\right)_y \\ \left(\bar{B}_0^{\perp} + \bar{B}_1^{\perp}\right)_x = \left(\bar{B}_2^{\perp}\right)_x \end{cases}$$

Why there is „–" ?

$$\begin{cases} E_0^{\parallel}\cos\varphi - E_1^{\parallel}\cos\varphi = E_2^{\parallel}\cos\psi \longrightarrow x \\ E_0^{\perp} + E_1^{\perp} = E_2^{\perp} \longrightarrow y \\ \frac{n_1}{c}E_0^{\parallel} + \frac{n_1}{c}E_1^{\parallel} = \frac{n_2}{c}E_2^{\parallel} \\ \frac{n_1}{c}E_0^{\perp}\cos\varphi - \frac{n_1}{c}\cos\varphi\, E_1^{\perp} = \frac{n_2}{c}\cos\psi\, E_2^{\perp} \end{cases}$$

from $\bar{E}$ border conditions

from $\bar{B}$ border condition

Please solve by Kramer's method

$$\Downarrow$$

$$E_1^{\perp} = \frac{n_1\cos\varphi - n_2\cos\psi}{n_1\cos\varphi + n_2\cos\psi} E_0^{\perp}$$

$$\bar{E}_1^{\parallel} = \frac{n_2 \cos\varphi - n_1 \cos\psi}{n_1 \cos\varphi + n_2 \cos\psi} E_0^{\parallel}$$

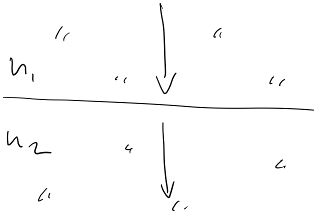
Fresnel's formulas

$$\bar{E}_2^{\perp} = \frac{2n_1 \cos\varphi}{n_1 \cos\varphi + n_2 \cos\psi} E_0^{\perp}$$

$$\bar{E}_2^{\parallel} = \frac{2n_1 \cos\varphi}{n_2 \cos\varphi + n_1 \cos\psi} E_0^{\parallel}$$

Let's see different cases.

① Normal incidence.



Since the incident plane can be any, then $\parallel$ and $\perp$ components are the same. $\varphi = \psi$

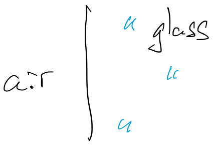
$$E_1 = \frac{n_1 - n_2}{n_1 + n_2} E_0 \qquad E_2 = \frac{2n_1}{n_1 + n_2} E_0$$

For example, the boundary between air and glass

$n_1 = 1$

$n_2 = 1.5$

$$\Rightarrow \quad E_1 = -\left|\frac{n_1 - n_2}{n_1 + n_2}\right| E_0$$



$$E_1 = e^{i\pi} E_0 \cdot \mathrm{Const}$$

When the light is reflected from more dense medium, then we lose half a wavelength, i.e. phase shift π.

If it reflects from less dense medium, nothing will change.

Why for $E^{\parallel}$ the sign will change.

Since $I \sim E^2 \Rightarrow R = \frac{I_1}{I_0} = \frac{(n_1 - n_2)^2}{(n_1 + n_2)^2}$.

R - reflection coefficient.

Eyes of invisible man problem.

Glass in air, water and glycerol.

② Tilted incidence.

Since $\frac{\sin\varphi}{\sin\psi} = \frac{n_2}{n_1}$

$$\begin{cases} E_1^{\perp} = -\dfrac{\sin(\varphi-\psi)}{\sin(\varphi+\psi)} E_0^{\perp} & \text{Why "$-$"?} \\[2ex] E_1^{\parallel} = \dfrac{\operatorname{tg}(\varphi-\psi)}{\operatorname{tg}(\varphi+\psi)} E_0^{\parallel} & \text{Look at tg} \\[2ex] E_2^{\perp} = \dfrac{2\cos\varphi \sin\psi}{\sin(\varphi+\psi)} E_0^{\perp} \\[2ex] E_2^{\parallel} = \dfrac{2\cos\varphi \sin\psi}{\sin(\varphi+\psi)\cos(\varphi-\psi)} E_0^{\parallel} \end{cases}$$

We assume air-glass border, i.e. $n_1 = 1$, $n_2 = 1.5$

First, we look at $E_1^\perp$ and $E_0^\perp$

a) $\varphi \to 0$

$$\frac{E_1^\perp}{E_0^\perp} = \frac{1-1.5}{1+1.5} = -0.2$$

b) $\varphi \to \frac{\pi}{2}$

$$\frac{E_1^\perp}{E_0^\perp} = -1$$

Now, we look at $E_0^\parallel$ and $E_1^\perp$

$$E_1^\parallel = \frac{tg(\varphi-\psi)}{tg(\varphi+\psi)} E_0^\parallel$$

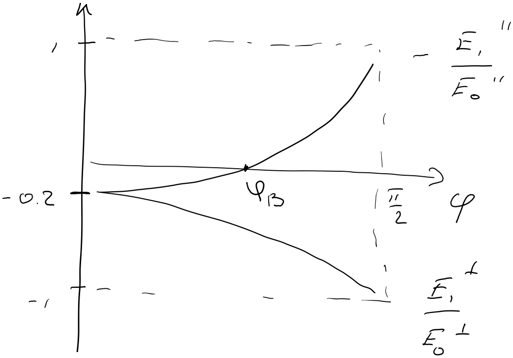
tg behaves wierd

$\varphi+\psi = \frac{\pi}{2}$ $tg \to \infty$

$\varphi+\psi > \frac{\pi}{2}$ $tg \to$ „$-$"

1) if $\varphi+\psi < \frac{\pi}{2} \Rightarrow \frac{E_1^\parallel}{E_0^\parallel} > 0$

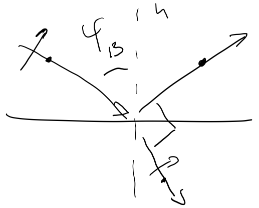
2) if $\varphi+\psi = \frac{\pi}{2} \Rightarrow \frac{E_1^\parallel}{E_0^\parallel} = 0$



φ_B - Brewster angle

$$\varphi_B + \psi = \frac{\pi}{2}$$

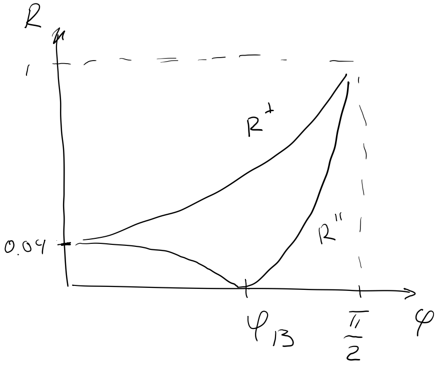
$$\frac{\sin\varphi_B}{\sin(\frac{\pi}{2}-\varphi_B)} = tg\,\varphi_B =$$



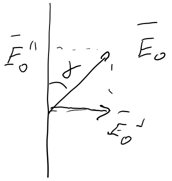
$= \frac{n_2}{n_1}$

For reflection:

$$\begin{cases} R_{\parallel} = \left(\dfrac{E_1^{\parallel}}{E_0^{\parallel}}\right)^2 \\ R_{\perp} = \left(\dfrac{E_1^{\perp}}{E_0^{\perp}}\right)^2 \end{cases} \Rightarrow$$



Let's now imagine polarization of $\bar{E}_0$ is at angle γ with the incident plane.



$$\begin{cases} E_0^{\perp} = E_0 \sin\gamma \\ E_0^{\parallel} = E_0 \cos\gamma \end{cases} \Rightarrow R_\gamma = \left(\frac{\bar{E}_1}{\bar{E}_0}\right)^2 =$$

$$= \frac{(E_1^{\parallel})^2 + (E_1^{\perp})^2}{E_0^2} = \frac{R_{\parallel}(E_0^{\parallel})^2 + R_{\perp}(E_0^{\perp})^2}{E_0^2}$$

$$R_\gamma = R_{\parallel}\cos^2\gamma + R_{\perp}\sin^2\gamma$$

When one use natural light with chaotic polarization, then γ is averaged.

$$R = \frac{1}{2}(R_{\parallel} + R_{\perp})$$

Natural light, when reflected is

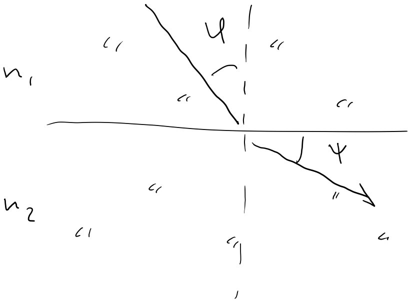
partially polarized.

Demonstrations

Letters behind the paper.

Glass pieces in water.

③ Full reflection



$n_1 > n_2$

$$\frac{\sin\varphi}{\sin\psi} = \frac{n_2}{n_1} < 1$$

$\psi > \varphi$

When $\psi = \frac{\pi}{2} \Rightarrow \varphi = \arcsin \frac{n_2}{n_1}$

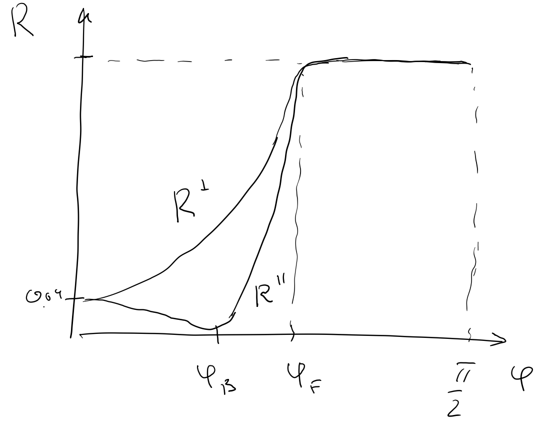
What does it mean for reflection?

$$R_\parallel = \frac{tg^2(\varphi_F - \psi)}{tg^2(\varphi_F + \psi)} = 1$$

φ_F

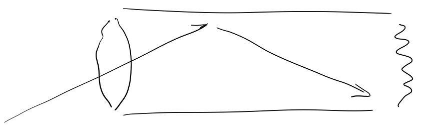
$R_\perp = 0$

$R_\perp = \ldots = 1$

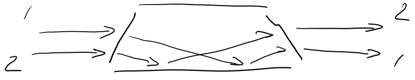


How do we use the effect of full reflection?

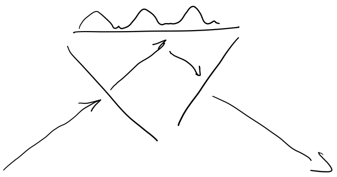
1) Optical fiber.



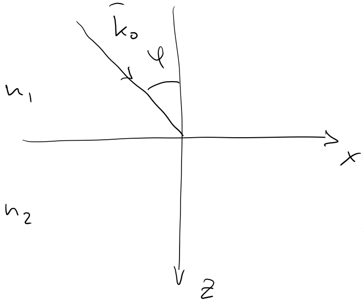
2) Prism



3) Total attenuated reflection



Let's carefully examine refracted wave.



$$\bar{E}_2 = \bar{E}_{20}\, e^{i(\omega t - \bar{k}\bar{r})} =$$

$$= \vec{E}_{20}\, e^{i(\omega t - k_{2x} x - k_{2z} z)}$$

$$k_2\bar{r} = k_{2x} x + k_{2z} z \qquad k_{2x} = k_{0x}$$

$$k_{2z} = \pm\sqrt{k_2^2 - k_{2x}^2} = \pm\sqrt{k_2^2 - k_0^2 \sin^2\varphi}$$

Since $k = \frac{\omega}{c} n$

$$k_{2z} = \sqrt{\left(\frac{\omega}{c} n_2\right)^2 - \left(\frac{\omega}{c} n_1 \sin\varphi\right)^2} = \pm\frac{\omega}{c}\sqrt{n_2^2 - (n_1 \sin\varphi)^2}$$

$$\boxed{n_1 \sin\varphi > n_2} \Leftarrow \varphi > \varphi_F$$

$$k_{2z} = \pm\frac{\omega}{c}\sqrt{n_2^2 - n_1^2 \sin^2\varphi} = \pm i\frac{\omega}{c}\sqrt{n_1^2 \sin^2\varphi - n_2^2}$$

$$\bar{E}_2 = \bar{E}_{20}\, e^{i(\omega t - k_{2x} x)}\, e^{\pm\frac{\omega z}{c}\sqrt{n_1^2 \sin^2\varphi - n_2^2}}$$

„+" contradicts energy conservation law

Let's call $z_0 = \dfrac{c}{\omega\sqrt{n_1^2 \sin^2\varphi - n_2^2}}$

$$\boxed{\bar{E}_2 = \bar{E}_{20}\, e^{-\frac{z}{z_0}}\, e^{i(\omega t - k_{2x} x)}}$$

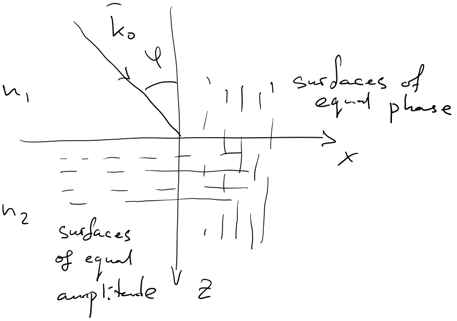
Evanescent field

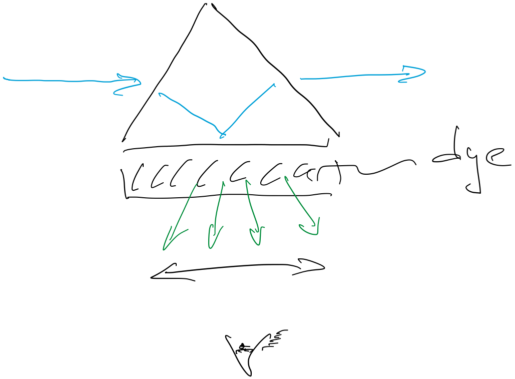
z_0 is the distance, when amplitude will drop e times.

For example, glass – air interface

$$z_0 = \frac{\lambda}{2\pi\sqrt{n_1^2 \sin^2\varphi - n_2}}$$

$$z_0 \sim \frac{\lambda}{2}$$



Mandelshtam - Zeleni experiment